An Exploration in the Space of Mathematics Educations

By Seymour Papert

This article appeared in the *International Journal of Computers for Mathematical Learning*, Vol. 1, No. 1, pp. 95-123, in 1996.

Metaphorical Intentions

A mathematical metaphor frames the intentions of this paper. Imagine that we know how to construct an N-dimensional space, ME, in which each point represents an alternative mathematics education -- or ame -- and each dimension a feature such as a component of content, a pedagogical method, a theoretical or ideological position. Each "reform" of mathematics education introduces new points and each fundamental idea a new dimension. Thus, if one considers a particular point (an ame) in ME, among its many "coordinates" are a (metaphorical) measure that runs from informal to formal and another that runs from instructionist to constructivist. In the paper I shall define seven more such oppositional principles that have not been recognized in the past as structuring choices in mathematics education.

Within ME I distinguish two subsets. SME (S for school) is the set of mathematics educations to be found on any significant scale in schools; DME (D for defensible) is the set of ames that could be defended as serving the social and mathematical purposes that justify the expense and effort of education in mathematics.

Although a hard-nosed practical educator might frown on spending time and money designing an ame with any intention other than implementation, the exploration of ME has a theoretical interest that transcends immediate applicability of any individual ame. The most down-to-earth intention of this paper is to convey a sense of a particular ame, called *z*, which is presented here not as a proposal for school reform but rather as an exercise in "pure" research in mathematics education. A thought experiment. With ten more in the same spirit we would begin to have an idea of the space we inhabit as theorists of mathematics education.

One could conceptualize the exercise of constructing ames as giving serious technical content to the discussion (usually conducted during cocktail hour) of the limits of what counts as mathematics education. For example one might wonder whether there exists an ame in DME whose set of topics is disjoint from school mathematics: no number crunching skills, no algebra (at least in the school sense of that word) and no Euclid. Most mathematics educators, in fact most people, would be skeptical. But how can anyone really know without investing the time, effort and mathematical creativity needed to carry the construction of hypothetical ames far enough to serve as counter-examples to the conservative negative answer or as existence proofs for a positive answer? Indeed, how could a discourse emerge for talking and thinking about the real potential directions of development of mathematics education except through the exercise of serious discussion of a diverse range of hypothetical directions?

The exploration of ME might confirm or refute what are for me two compelling theoretical intuitions. The first of these is a tantalizing sense of past thinking about mathematics education as being confined to a tiny subset of ME, thus creating for a next generation the exciting prospect of breaking into new worlds. The second is more fundamental and more relevant to immediate concerns of researchers: Small or big, SME's shape must be as obscure for us as a pentagram is for flatlanders unless we learn to situate it in its containing space. A step in this direction is identifying my seven new axes (or dimensions) of variation in mathematics education analogous to the formal-informal and instructionist-constructivist dimensions. Since these are the axes along which *z* is most clearly separated from SME, recognizing them gives structure to *z*'s placement in ME.

I have worked on and off for over thirty years on the construction of *z*. Some products of this activity have spun off and been assimilated (sometimes with positive and sometimes with negative valence) by the culture of school mathematics: notably the concept of *turtle* as a basis for a topic in elementary geometry, the concept of *microworld* as the context for project-oriented work and the concept of *Logo* as a programming language designed for learning. This assimilation could hardly happen without denaturing ideas developed in a perspective of more radical change to which this paper returns them. Public discussion of the original enterprise feels more timely and more realistic today because the accumulation of relevant ideas and experience has (though just barely) reached a needed critical mass; because *z*'s intensive use of technology is beginning to lose the aura of science fiction that impeded serious consideration at earlier times; and because a growing, even if still inchoate movement for radical educational change, is beginning to come out of the closet.

The presentation of *z* and the formulation of the oppositional principles will contribute to some currently active issues in the literature. In particular: (i) I see a theme of my work that I refer to here as "the principle of thingness" as an earthier cousin of the concept of *reification* which has become very lively in the past few years (Confrey, in press; Dubinsky, 1992; G. Harel, 1992; Sfard, 1992, 1994); (ii) a discussion of work on probabilistic thinking stands in here for a generally jaundiced view of what mathematics educators can learn from the experimental cognitive studies paradigm (Kahneman & Tversky, 1982; Shaughnessy, 1992; Wilensky, 1993, 1995); (iii) the paper as a whole is intended to broaden discussion of the role of computers in the evolution of mathematics education by suggesting a more transformative impact on changes in content then is suggested even in the most forward-looking publications in the mathematics education literature (diSessa et al., 1966; Kaput, 1992; Noss & Hoyles, 1996).

The design of *z* is guided by (i) wanting it to be as *different* as possible from SME in identifiable, theoretically interesting ways (so that far reaches of ME can be probed); and (ii) wanting it to use ideas that can be seen as extrapolations from actual experiences of real people working at mathematics in forms at least piece-wise similar to *z* (so that we can appeal to intuition to inform judgment about *z*'s plausibility and mathematical quality).

In the rest of the paper I present a series of spots intended to show aspects of the kind of thinking that informs *z* applied to levels ranging from pre-school children to mathematically educated adults.

1. A First Peep at *z*

I start with a corner of *z* that is likely to have immediate name recognition as an idea that's in the air: bringing probability into elementary mathematics. We peep (in an imaginary but only slight extrapolation from real conditions) at children as young as five or six using an iconic form of MicroWorlds Logo to create animations and simple green games. Specifically we observe one child leaning over to ask another, "How did you make your colors do *THAT*?" and being shown a screen object whose name is RANDOM and whose simplest behaviors can be grasped in minutes.

The first of my oppositional principles is brought out by contrasting the activity of these children with a more common way to introduce probability in schools by using physical materials such as spinners or dice to introduce children to the idea of probability. I ask: what can these children do with this new knowledge besides talk or deal with teacher-initiated problems? How can a child actually use it to do something that has real personal importance *now*? My Logo kids are excited because they can produce dramatic screen effects and will go on using their growing control over random processes in projects of increasing complexity. What can your spinner spinners do that will give them a sense of empowerment and achievement?

The principle is called the power principle or "what comes first, using it or 'getting it'?" The natural mode of acquiring most knowledge is through use leading to progressively deepening understanding. Only in school, and especially in SME is this order systematically inverted. The power principle re-inverts the inversion.

As part of a study of the perception of the principle in education circles, I asked a group of teachers to rate the relationship between a spinner and RANDOM on a scale from "no relation" to "no difference." I was surprised to find that the high school math teachers in my admittedly too small sample of six saw less difference and made more comments like "both illustrate the concept of probability" than the (non-specialist) elementary school teachers who saw more difference and were more prone to make comments like "you can do more with RANDOM." It seems likely that these answers reflect a difference between the culture of elementary and high schools. In the former, project-oriented work may not be dominant, but it is sufficiently present to make teachers think about new ideas as a source of power to do something. In the latter, the emphasis is not on *using* ideas but on *understanding* them. The spirit of *z* is more akin to that of the elementary school, though it goes very much further in using the power principle as a criterion for what is taught as well as how it is learned.

The use of the power principle in *z* is closely related to two other inversions: project/problem and media/content. The girl who asked her friend about her color effects certainly had a problem and found a solution. But had you asked her what she was doing she would not have said "problem solving." She may have said "working on my flowers project" and she would have gotten the order right: projects are primary, problems come up in the course of projects and are sometimes "solved" and sometimes "dissolved." It is an inversion order to define the goal of mathematics as problem-solving (though this is better than defining it as rote learning of multiplication tables), and the design of *z* inverts this inversion too in the form of the principle "project before problem." Undoubtedly some people like doing problems that are not parts of projects and this is their right. I imagine I would be counted as one of them, although I suspect that my joy in solving apparently dissociated problems is part of a larger project I don't know how to name. But in any case the solving of an isolated problem just because someone asked you might be fun but it is not what mathematics -- or life -- is about.

The power principle can be carried far because *z* assumes excellent material and cultural conditions for working on mathematically rich projects and this in turn rests on another shift. The typical activity in contemporary school is making inscriptions on paper. The typical activity of *z* is manipulating a computer-based microworld. Of course it is not to be assumed that the shift of media has radical consequences in itself. As long as it is assumed that content primes over media, the new media will be used to support the old content and will often do this badly since the content was defined for the old media. The relevant inversion that informs the design of *z* leads to assuming that new media open the door to new contents.

2. Re-empowering Probability: A Sketch of the Stochastic Presence in *z*

The little peep at children using RANDOM in their projects showed the presence of a stochastic idea but did not show anything resembling the ideas that would normally be included in a school course on probability. This is the result of a deliberate judgment that the best use of the new media would be achieved by taking *random variable* rather than *probability* as the introductory concept for a strand of *z* that should then be called something more general than "Probability," such as "probabilistic thinking" (abbreviated here as PT) or "stochastics."

This might seem perverse since the idea of random variable would generally be regarded as mathematically more sophisticated than probability, which can, after all, be introduced using mathematics as simple as fractions. One might fear that in the end the use of RANDOM would turn out to be pragmatically powerful but mathematically weak. Had this been the case I would have turned it over to teachers of art or biology and not been writing about it here. However it is not the case. RANDOM can be presented in a way that is powerful not only for the applied results it achieves but also for the richness of its connections with mathematical ideas.

When I was comparing RANDOM with the spinner I focused on its greater pragmatic power. Later in this section we'll see even more compelling examples to show how *z* can give children at ages like 6 or 8 or 10 a sense of the power of probabilistic thinking that far surpasses what graduate students in education schools routinely get from their courses on probability and statistics. This is what qualifies it as empowering. The use of

Re-empowering in the heading of this section reflects a hope of reversing a historical process through which one of the most powerful ideas in our intellectual heritage is (not untypically) disempowered in its school presentation, where it is reduced to shallow manipulations that seldom connect to anything the student experiences as important.

The degree to which RANDOM has greater pragmatic power than the spinner is impressive. But as a mathematician I see another difference as more fundamental and another limitation in the thinking of the math teachers I interviewed as much more saddening. In their ratings of degrees of difference, none of them referred at all to the idea that the spinner is a physical object while RANDOM is a mathematical object. Indeed in conversation it was clear that the idea of a mathematical object was very far from the center of their intellectual radar screens.

I shall go from here in two directions, taking the simpler first and the more subtle later, even though it is more fundamental.

(i) Let's note that RANDOM is sufficiently like a function[1] to be experienced by young students as such if the core of being a function is taken not as some abstract definition but as how functions are used, how they combine, what purpose they serve. RANDOM is used in functional compositions even at the most elementary level in instructions like

SETCOLOR RANDOM [blue green][2]

I have seen children just a few years older using RANDOM in an explicitly functional form in instructions like the following two ways to generate random "century numbers" up to a million:

100*(RANDOM 10,000)

WORD (RANDOM 10,000) "00[3]

These nine- or ten-year-old children are very comfortable with making a new function they might call, say, RANDOM-CENTURY using Logo methods of procedural definition. This goes way beyond any traditional transaction between mid-elementary school students and functions and is presumably far short of what the *z* students would do with their greater opportunities to develop computer fluency.

(ii) In (i) we saw RANDOM as a carrier of the idea of function. It can be a carrier for many other mathematical ideas and perhaps most importantly for the idea of a mathematical object. As I develop a first insight into this I shall be showing how to follow yet another of the seven principles, the *thingness principle*: object before operation.

Let me stand back and show how I see this principle applying at even earlier ages, even before the child feels comfortable with typing and reading. It is common today to use computer interfaces that replace typing instructions by combinations of pointing, clicking, dragging, pull-down menus and so on and so forth. I think this is overdone and often done badly[4]. A significant step toward a good use is shown by MicroChild, a version of MicroWorlds Logo made by A. Sopranov[5] in which Logo-like instructions and even procedures are constructed by putting "blocks" together, like making a Lego house. In this context, with a little more development, ideas of function, composition of functions, definition of functions, transformation of functions can become perfectly natural to preschool children. More important, they would be accustomed to thinking of functions (and of course many other mathematical entities) as thing-like: you put them somewhere, you move them from one place to another, you build with them, you give them names . . . all this I see as making "thingness" operational by giving entities thing-like properties and by relating to them as things.

What can be done at such early ages can be done at later ages. In a while I shall develop an example in which entities like gravity are given a thing-like representation not to avoid typing but to help thinking. In much the same way that object-oriented programming facilitates the development of software, object-oriented theories can facilitate the development of theories by a beginner (Papert, 1992).

More Powerful Work in PT

I have worked even more extensively with children in the eight-to-ten age range on another kind of problem where probabilistic thinking is experienced as a powerful way to get things done.

Consider a hypothetical student who has constructed a simulation of tropism by programming a screen creature (called a logomecium) to move toward another screen object (the goal). The logomecium is able to detect whether the goal is more to the left or to the right and its law of motion is to move forward continuously making small adjustments of heading in the direction detected. This kind of construction comes up naturally in a variety of contexts including biological simulations (from which the logomecium gets its name), games (for example when one character chases another) and some interesting geometric situations (e.g. if the goal follows a path, say a circle, what does the logomecium do?).

So far no PT, but at least an insight into another strand of *z* I like to call "cybernetics" that includes building physical as well as screen-based "creatures." *But probabilistic thinking comes into the picture when we consider what happens if the logomecium encounters an uncrossable (but transparent) obstacle.* The solution that is easiest to implement and can also lay claim to being conceptually the richest consists of introducing a probabilistic element into the behavior of the logomecium. The implementation is made especially perspicuous (and easy) in MicroWorlds Logo by its capacity for multi-processing. The original program can be left intact -- and even running -- while a randomizer agent is created as a separate entity for example to make the logomecium turn randomly every so often.[6]

Students naturally ask: Is this a better behavior? This is a good theme for discussion: the randomness makes the logomecium a little less efficient if there is no obstacle and much more efficient if there are some. Many rich issues, some on the fringes of statistical inference, are raised by attempting an experimental comparison of several logomecia -- with and without the randomizer and using different algorithms in the systematic or in the random components of behavior. In my experience of similar situations (I have not actually worked with this one), the students will begin by thinking that the question could be settled by a race, that is to say, by one run of a race. However, if the contestant creatures were created by members of the group, the losers have a sufficient stake in a critical stance for heated discussion, out of which can emerge the idea that more that one run is needed to make any inference and that the comparison of logomecia cannot be absolute since A can be superior (statistically) to B with a certain configuration of obstacles while B beats A with another configuration.[7]

The logomecium's behavior using the random strategy can fruitfully be compared with the computationally more complex behavior of recognizing that it is blocked (which it does not have to do for the random strategy to work) and finding its way around the obstacle. The idea that obstacle avoidance can have probabilistic and/or deterministic components leads to looking at real creatures (e.g. paramecia or flies or bees) and trying to determine whether there is a probabilistic element in their behavior. Whatever the outcome, there is room here for discussion about the use of randomness as a powerful problem-solving technique. It is not hard to elicit animated discussion about the many ways in which "nature" has "used" this technique. A teacher (or a text or an advice program) could encourage individual students to undertake research projects aimed at searching for situations where randomness is useful as well as for situations in which randomness is a nuisance to be overcome. In the next section we meet a case of using randomness in a mathematical context

Monte Carlo

A powerful and very general use of random process is seen in the "Monte Carlo" method of finding the area of a region by the analog of throwing darts and calculating the ratio of hits to misses. The graphs in Figure 1 represent on the *y*-axis the ratio of hits to misses for a circle and its inscribed square against the number of throws sampled every 50 times for a total of 20,000. They were made in the course of exploring questions about how many "darts" to throw and helped answer that question, but also gave rise to a series of mathematical questions that flourished into a full-fledged mathematical investigation.

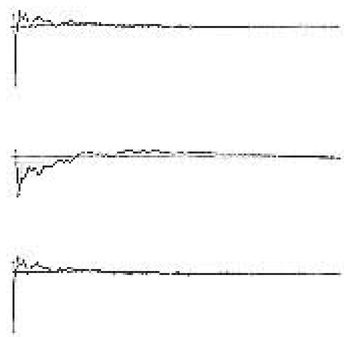


Figure 1

In many similar situations I have had excellent conversations with middle school students and some younger ones, based on looking at these graphs in more detail than simply noting their convergence to an asymptote. Questions that come up often are about why it stays so long on one side and indeed about why it converges at all. One group of middle school students spontaneously raised the question: "If after 10,000 shots a freak run took the graph far off, would it come back again faster or slower than in the early part?" They thought it would be faster. In what I see as a prime example of the value for young mathematical investigators of being fluent in Logo, it took less than 30 minutes to mount an experiment that allowed perturbations to be introduced by "forced runs."[8] Figure 2 shows what these kids interpreted with amazement as recovery being slower from perturbations that come late in the process. But they also found that it was "harder" to produce "the same" perturbation. And this led them to engage with ideas that eventually led to inventing an informal but perspicuous proof of a form of the law of large numbers.

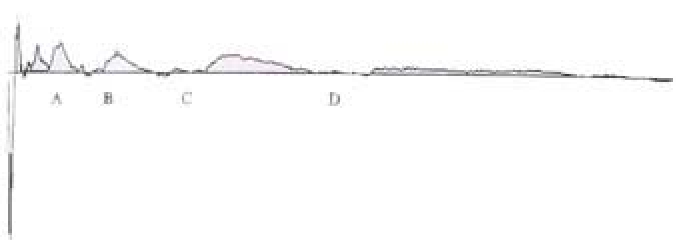


Figure 2

The Power of Project Immersion

In case the last paragraph seems to be too good to be true, it is time for a digression on the beginnings of an accumulation of data that show people performing at greatly superior levels when they are immersed in projects.

A particularly clean example is an apparent paradox in the report by Sfard and Leron (in press). They ask which of the following two problems is harder:

P1: Given three points, (2,3), (-1,4), and (0,-1), in the plane, find the center and the radius of the circle through them.
P2: Write a computer program that accepts any three points in the plane (given by their coordinates) and returns the center and the radius of the circle through them.

Since P2 asks for more than P1, one could argue that tautologically it must be harder. But Sfard and Leron report that more than half of their students failed at P1 and nearly all succeeded at P2. Why? It is not any magic of the computer except the fact that working at the computer transforms the stance of the students from "solving a problem" to "pursuing a project."

Something similar happens when a transition is made from "thinking about" the graphs in Figure 1 to an engagement with trying to transform the process. Uri Wilensky wrote a thesis (for which I had the great pleasure to be advisor) based on a brilliant series of case studies that show people (in his case adults of very varied degrees of sophistication) shifting from the perplexed denseness usually evoked by questions about probability to high-quality intellectual engagement when the questions became part of a "learner-owned investigation." Another of my thesis advisees, Idit Harel, shows how the performance of a mathematically poor student is turned around when she becomes deeply (and emotionally) involved in a multi-month project of software development.

Similar observations can be culled from working with students on fragments of the PT strand of *z*. As background to one of these I recall one of the many experiments by Kahneman and Tversky that have supported a prevailing view that "people" are inherently poor at probabilistic reasoning. In the "taxi problem," subjects are asked to give their estimate of the probability that the taxicab seen by a witness was blue or green given (i) the witness said it was blue (ii) independent tests show that the witness makes correct identifications 80% of the time and (iii) 85% of the cabs are blue. A surprisingly high proportion of subjects base their conclusion entirely on (i) and (ii) and ignore (iii). The following story offers insight into a kind of experience that might foster a better development of probabilistic judgment. The story also touches on the strength of making (rather than playing) computer games as a context for mathematical learning.

I have constructed collaboratively with eight-year-old children and successfully played games like the following.[9] There are two kinds of screen critters called reddies and greenies. They look exactly alike and each kind lays eggs whose color is determined by a random variable whose possible values are red and green. But in the two cases the random variable has different distributions: for example, looking inside the critters at their programs one might see RANDOM [red red red green] in a reddy and RANDOM [red green green green] in a greeny.

A critter appears and begins to lay eggs. The player has to guess whether it is a reddy or a greeny. Points are gained for correct guesses and lost for incorrect guesses. There is a penalty for taking time to make the guess; perhaps points are lost as a function of time or perhaps the penalty takes the form of allowing a second player to get in first with the guess.

Variants of the game have different consequences for making the error and for taking time. For example, suppose the player gets a point for guessing right and loses one for guessing wrong. In this variant one might as well guess immediately. But if the cost of a wrong guess is ten points and the cost of delaying the decision not too heavy, it is better to observe several eggs before deciding.

I certainly saw examples of the Kahneman and Tversky finding. However, I also saw another behavior in one child who was eager to set the game up in such a way as to "catch" the player. With this different *intention*, the idea of *exploiting the taxi problem fallacy* presented no difficulties whatsoever. Switching from a problem-solving stance to a project-construction stance radically transformed the "difficulty" of the idea of *a priori* probability. In terms of idea-power theory, the shift could be seen as changing the status of the *a priori* probability from an annoying factor that complicates thinking (and that one would just as well forget) to a useful, empowering idea that one can use to get a desired result. Otherwise put: appropriating the fallacy for personal use led quickly to understanding it.

In my view this leads to a radically different interpretation of the Kahneman-Tversky findings: people whose relationship with probabilistic (or any other) ideas has always been superficial use this thinking superficially. Learning probability by throwing dice and calculating (ugh!) fractions will reinforce what Kahneman and Tversky find. Integrating it as a powerful and empowering idea will give rise to something else.

3. Kinetic Microworlds: Dynamics Before Statistics

The computational context of PT illustrates another of the seven principles, which I'll state here as *dynamics before statics* using this formulation to contrast it with the tendency in school to place *statics before dynamics.*

I first became convinced of the importance of this issue through thinking about the almost universal pattern in school physics of placing statics before dynamics. This ordering appears odd, an inversion of the natural order, from a number of points of view. I start with the most fundamental.[10]

Classical physics is most coherently presented as deriving logically from a set of propositions aptly called Newton's *Laws of Motion* -- aptly because they are essentially principles of dynamics. However, the pedagogical order when elementary physics has any theoretical content comes close to a complete inversion of this logical order. The reasons for this given to me by a large number of science teachers in an informal survey could all be reduced to the idea that dynamics actually is conceptually intrinsically harder. Some of the more articulate ones amplified this by pointing out that one needs calculus to do dynamics seriously and calculus comes at the end of a prerequisite chain that runs something like arithmetic to algebra to calculus.

Now one can quibble in many ways about the necessity of this ordering, but I believe that the fundamental issue is that there is truth in it insofar as it reflects the static nature of pre-computational media. To state a complex matter far too simply, calculus is a way of representing dynamic phenomena in the static medium of pencil and paper; it is "hard" because the medium fights the message.

The theme of this section is to show some of the less obvious ways in which new dynamic media, illustrated by a strand of work in *z* which I'll call "dynamic thinking" and abbreviated by DT, allow us to invert the inversion.

The work with videogames and logomecia contributes to this by introducing young students to a mindset for which the phrase "law of motion" evokes a warm and natural connotation rather than the fuzzy bewilderment that surrounds the idea when the first example encountered is Newton's. Surely this is a step toward learning physics as well as mathematics. But what comes next?

3a. Object-oriented Study of the Study of Motion

A good example is the empowerment of the idea of decomposition of motions by starting off with the use of a tool for the construction of useful motions. Cast in this form, the idea has a seed no more difficult at age six than Etch-a-Sketch. A simple point-and-click tool allows the combination of horizontal and vertical motion to produce motion in any desired direction. Once this is firmly in place intuitively it is an easy step to use a tool that will split velocities into components that need not be parallel to the axes.

Following the principle of thingness, I believe that the best way to do this involves "thingifying" motions (or perhaps velocities). Thus if we are making a game and wish to create a character we shall be moving in complex ways, it would be a good idea to create an icon for the character's motion. We can then *name the motion*, place it in a corner of the screen, move it over to a tool in order to change it, split it, combine it with other motions or whatever. In other words it can behave like a thing and so should be counted as a thing. I have no doubt that thingifying motion makes it easier to work with and ultimately to understand.

Another powerful example of thingification is seen in the following detail. A student is trying to make a videogame in the style of Mario Bros. where a character runs across the screen encountering various challenges and obstacles. It is natural to want to make the character jump, but what is a jump? How does jumping work?

The obvious simple way to make a jump is to give it a rectangular path. In traditional Logo one might do SETY YCOR + 1 and later SETY YCOR -10. But most of the students with whom I have worked are not satisfied with this jump. To obtain a more realistic, or at least a more interesting, trajectory, several powerful ideas are helpful.

The first of these is naming (which really means thingifying) the mathematical object we call a trajectory. This is not something one can see, and in my experience children do not think of the path as a thing with its own identity. The next powerful idea is combining velocities to make new velocities. When this is available the character in the game might be represented as having not one but two velocities, a horizontal velocity and a vertical velocity. Suppose at the beginning that both are set to zero speed. One can make the character run by making the horizontal speed, say, 20. Now initiating a jump could start by making the vertical speed 10. This is a step in the right direction but results in the character going off into the sky.

I have not yet seen a youngster who does not immediately say something like, "We need gravity." The next problem is how to represent gravity. Again there are many approaches; I have mostly chosen to point students toward what seems to me the most powerful approach: thinking of gravity as a thing and representing it as a computational object.

Thus gravity comes to be represented by one of the more powerful modern computational ideas: *an object inherently endowed with the capacity to have properties*. As a first shot at specifying those properties I have found it useful to suggest that "gravity eats vertical velocity." Whenever another object has a vertical velocity, Gravity, now personalized, detects this and begins to nibble away at it. So our figure's vertical velocity will successively be 10, 9, 8 and eventually 0 and then -1, -2, and this means that it is coming down. But there is still a bug. When the character hits the ground it should stop, but on the present definition of gravity it will just keep going into the ground. We next have to build into the definition of the gravity object a clause that turns it into: gravity eats vertical velocity of *unsupported* objects. When I have worked with children they have generally been content to use being over the ground's color as the criterion to judge "being supported." But in some cases this criterion was seen as inadequate. This does not mean it was seen as "wrong" but rather as needing more work. So we see an evolutionary process. At each stage there is a working system which manner of working is seen as flawed, but on examination gives suggestions about how to change it to be closer to what is desired.

I conclude this section with the observation that this representation of gravity as a computational object is so different from standard representations that it deserves a name; I think that *"object-oriented theory"* would be appropriate. And so we see, to make a point that deserved more discussion than it is getting in this paper, mathematics education can draw on computation not only to provide physical computers, but to provide a much richer set of new representations of knowledge than the idea of "procedural representation" that has slipped into cognitive discourse.

3b. The Parabola

A mathematician should not be satisfied with simply making the jump. One would also like to know about the properties of the trajectory of the jumper, in particular that it is a parabola. But what does this mean?

In the narrow school perspective this would mean proving that "the equation of the trajectory is $y = ax^2 + bx + c$." In the perspective of designing *z* I take a different approach by questioning the privileged role given by SME to that particular "definition" of parabola. I conjecture that this shows the influence of an educational technology of pencil and paper that makes plotting curves from equations an exercise that is both easy to prescribe and easy to grade. In the perspective of the kind of work we have been looking at one might instead take as the first formal characterization of a parabola the following: the trajectory followed when the *x* velocity and the *y* acceleration are both constant. Formally this is strictly equivalent to SME's and is at least as supportive of investigation of properties of parabolas!

My point here is not to argue that one characterization of the parabola is better than another, but only to give what I see as a good example of how the influence of the media used can make one or another seem to be the natural one. But whichever is better when one looks at the isolated case of the parabola, there is no doubt that in general much more can be done at an elementary level with dynamic than with algebraic characterizations of curves. Think for example of how easy it is to work with spirals in turtle geometry or with computer generation of fractals.

4. Turtle Geometry

I have refrained from explaining what kind of formalism is used to produce and control the various animations and movements used in PT and DT because it is orthogonal to my purpose: this could be done in many ways to achieve the same effects. However, the best way in my opinion is still through the use of turtle geometry, and this is what I would use as the foundation for a personal version of *z*.

I assume that the reader is familiar with the turtle concept, but a few remarks may be necessary here to ensure that it is understood in the same form. (In section 6, below, a more advanced turtle topic than usual will be developed).

Turtle geometry -- the name and the concept -- has been used in a variety of ways. In a form more often called turtle graphics, it is frequently used as a programming topic for beginners, mainly but not only in Logo. Abelson and diSessa have drawn wide attention to the idea of using it as a topic that lends itself to relatively advanced mathematical explorations. Noss and Hoyles (1996) have most directly recognized the value of turtle geometry as a research tool to study the development of mathematical thinking, using it as a "window into the mind" (see also the work of Harel, 1991; Lawler, 1981,1985; Resnick, 1994; Solomon, 1986; Weir, 1986). A number of authors have used it as a tool for producing attractive graphics and for gaining insights into graphical pattern structures. These studies each exploit a partial aspect, but lost the holistic view, of the motivating vision of a well-rounded piece of mathematics that would allow simplified entry points for very young novices and yet have multiple connections with mathematical ideas.

Consider for example how the idea of *theorem* can be re-empowered for a student in the context of actually working with turtles. In *z*, as in working with children, I favor introducing the word *theorem* at a very early age usually instantiated first by what I call "The Total Turtle Trip Theorem" (TTTT) presented in some such form as "If the turtle makes a trip back to its starting state without crossing its own path, the total turn is 360 degrees." This statement assumes prior definitions of "state" and the total turn (the algebraic sum of all its turns), but this is no burden since both of these are empowering concepts for someone trying to control turtle movements. And both are more generally powerful concepts as measured by the ramifications of their connections in mathematics and science.

As presented, what makes the TTTT a "theorem" is the possession of several attributes: first, it is powerful; second, it is surprising; third, it has a proof. And although what we can offer as a proof may not reach the standards of a formal logician, it is as sound as the typical style of proof of working mathematicians. Similarly, the advice "play turtle" given to someone trying to figure out how the turtle should move is not only very effective in solving the problem, it is also very effective in conveying the idea of "heuristic."

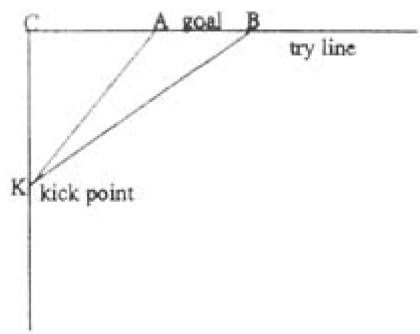
A final point I want to make here will be amplified in the next two sections. This is the wide range of kinds of mathematical work that can be done with the turtle. Consider for example the problem of finding a formula for the area of a circle. I did not mention the word "turtle," but the Monte Carlo procedure made typical use of turtle geometry not only by placing a turtle at a random position and testing whether it was in the circle but also for drawing the graph. But if one can only understand something one way, one has not understood it at all, and so it is relevant that turtle geometry lends itself to other interesting ways to look at this problem. For example, even at the most elementary level the circumference of the circle is seen as a limit of polygons, and so the area is easily represented as a limit of sums of triangles.

5. Rugby Session

Looking at a problem-solving session of which the participants are adult researchers in mathematics education allows me to discuss the spirit of *z* on another level, by comparison with the way in which a set of highly competent mathematics educators participated in a real mathematical activity.

I have selected a problem that has recently been the subject of published commentaries by several researchers whose general philosophies of education are close enough to my own to engage specific dialogue and bring out how *z* differs from the shared position in the "Logo community" (Noss & Hoyles, 1996; Resnick, 1993; Wilensky, this issue). All these authors participated in a workshop the members of which were invited to use their preferred software tools to tackle the following problem (summarized here from Wilensky's description):

After a try has been scored [by touching-down at point C], the scoring team has the opportunity to gain further points by "kicking a conversion" [i.e. kicking the ball into the goal -- the segment AB in the figure]. The kick can be taken from anywhere on an imaginary line [line CK in the figure] that is perpendicular to the try line [line CB in the figure] and goes through the point where the try was scored [C in the figure]. Where should the kick be taken from to maximize the chance of a score? (In other words, where should we place the point K, so as to maximize the angle AKB?)



Of the three approaches that appear to have been followed at the workshop, the first used Cabri, therefore taking the question as a problem in the familiar school version of Euclid; a second which took it as a problem in calculus and trigonometry is not relevant here; the third, followed by Mitchel Resnick and Uri Wilensky used a version of Logo that allows many thousands of turtles. Wilensky describes their thinking as follows:

We imagined several rugby players standing in each patch along the perpendicular line, CK, with each player kicking thousands of balls in random directions. To find the best kicking point, we simply needed to determine which rugby player scored the most conversions. It was quite easy to write the StarLogo program that would implement this strategy.

The technology of StarLogo facilitated our seeing the problem as amenable to probabilistic methods. Indeed, the most important piece of mathematics that can be garnered from this example is not the numerical solution to the particular problem, but the transformation of the problem from one of calculus or geometry to one of discrete probability.

The fact that the Resnick/Wilensky process seemed surprising, as participants report that it did, says something significant (and not surprising) about the culture of school mathematics. I am sure that all the commentators and everyone present at the meeting were aware of the idea of Monte Carlo methods, but the comments indicate that this way of thinking is not salient enough, or does not feel natural enough, to leap out as a method to use. Hoyles and Noss in their book (1966) associate the choice of method with the software tools best known to the problem solver, so that one would expect Resnick and Wilensky to adopt this approach because of their relationship to StarLogo. But in fact the Monte Carlo solution does not need StarLogo and the use of StarLogo for this problem does not need Monte Carlo. A program in plain Logo takes a little longer but does as well, and I shall describe below a non-probabilistic approach to which StarLogo is no less well matched, and possibly even better matched, as the tool of choice. *This episode is not primarily about software tools. It is about the status of ideas on an empowerment/disempowerment dimension.* I see the whole episode as confirming my view of PT as having the status of a disempowered idea in the culture of mathematics education, and of *z* as making a needed contribution by restoring its power status.

My personal approach to the problem when I first encountered it mobilized an idea that was as "hot" for me at the time as probabilistic methods might have been for Resnick and Wilensky. It took the form of a "heuristic drive" to break my thinking out of the line L and to visualize the subtended angle over the whole plane. My sense of connection was to potential fields and to questions about visualizing them by devices such as drawing equipotential contour lines or drawing a three-dimensional surface over a two-dimensional field.

As it happened, giving the question, this orientation immediately merged into seeing the answer: the contour lines must be circles through A and B; the smaller circle represents the largest angle, so the answer is found by looking for the circle that just touches L. I would be happy if I could believe that students in *z* would go through this process just as I did (as well as recognizing the possibility of the Monte Carlo method). But asking what I really want them to learn leads me to split this process into two parts. The more important part is what led to the orientation, that is to say to activating the general field idea (including an awareness of the issue of visualization). Knowing specifically that the equiangle lines happen to be circles is to me less important. So I tried to put myself in the position of a student who had taken the first but not the second of these steps.

Now I should also like students who pose that question to think naturally of writing a program to visualize the fields. Their *z* experience would have highlighted both the general idea of looking for means to visualize situations and also the particular idea used here. In principle, StarLogo should be ideal for doing this as a program (which, incidentally, has no PT in it) to "tell" each point P (each "patch" in StarLogo language) to measure the angle APB and take on a color as a function of this angle. In practice, accidents about what has and has not been implemented in current versions of StarLogo made me feel that it would be simpler and more perspicuous to use MicroWorlds to write a Monte Carlo program using a probabilistic "dart throwing" metaphor. My Logo program instructed a turtle to place itself in a random position, randomly measure the angle subtended by AB at this position and take on a color as a function of the result.

```
towards "A
name heading "a
towards "B
name heading "b
setcolor (:a-:b)
```

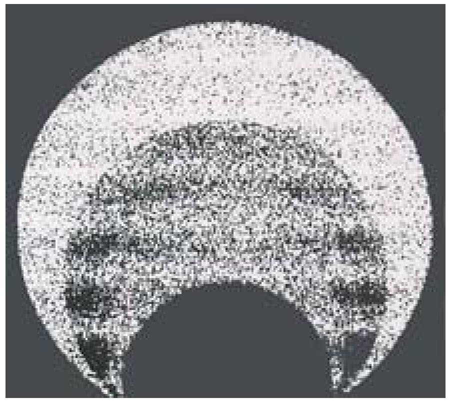


Figure 3

When I ran this procedure repeatedly the contour lines slowly emerged like a developing Polaroid photo. Only a few minutes were needed to recognize the emergence of circles; after ten minutes the graph was crisp and visually as well as conceptually pleasing.

Now the third point I want to make might be the most fundamental. Looking over all the approaches to the problem mentioned here (or in the other commentaries), I see the contribution of the computer as split into two kinds.

The Resnick/Wilensky solution is a pure case of the first of these: the computer allows the point to be found by actual measurement of the relevant quantity, namely the angle subtended at a point. The fact that ten thousand values can be measured in a way that feels to the user as simultaneous does not take away from the empirical nature of this method. The same would be true for marching the turtle down L using the just-cited fragment of Logo code to measure the angle and embedding it in a not-very-complicated superprocedure to find the maximum. The method is quintessentially computational but leaves one with an answer that is mathematically unexplained and poorly connected to geometrically related situations.

A pure example of the second kind is using Cabri as a far more powerful instrument than ruler and compass or free-hand drawings, to build the constructions typically used in thinking about traditional school geometry. The computer is used to find an answer in a pre?computational conceptual space.

In both cases the computer used as a tool effectively leads to a solution, but in neither does the computational representation make the mathematics more perspicuous.

The approach to geometric thinking that I have called *z* combines key features of both kinds of contribution. The goal is to use computational thinking to forge ideas that are at least as "explicative" as the Euclid-like constructions (and hopefully more so) but more accessible and more powerful. In the next section I illustrate the idea by using Turtle geometry to give the theorem about angles subtended by a chord greater perspicuity, a more intuitive proof and new connections to other ideas.

6. The Turtle Meets Euclid

I have always had a somewhat complex relationship with the subtended angle theorem. I have never been able to clear my mind completely of a feeling that there is something odd about that angle being the same as the point P moves along the circle. ("Surely down there in the corner between the circle and the chord the angle should be squished or something.") I would conjecture that such feelings could inhibit some people from thinking of this theorem and do the opposite for others. But in any case I feel that neither the standard Euclid-like proof nor playing with Cabri makes it transparently perspicuous that there should be a connection between the circleness of

P's path and the constancy of the subtended angle.

Looking at the situation in the perspective of Turtle geometry gives an insight into the connection and gives rise to a shorter proof (Papert,1986). Try seeing it in the following way:

In Turtle geometry the most basic images of the circle and of the subtended angle are quite close to one another. The primary way to think about the circle is a path of a turtle that turns as it moves forward. So the arc MPN is firmly associated with a certain amount of turn. Now think of the turtle trip that goes direct (in a straight line) from M to P, turns and then goes direct to N. Both set out from the same point, went through a motion involving turning and ended up at the same point. It seems natural to ask how the turnings in these two turtle trips are related. To do this, use a standard widely useful TURTLE GEOMETRY heuristic of having two turtles, T1 and T2, start in the same state (the same place and the same heading) and compare them as they do their trips.

So as Tl goes along the circle from M toward P, think of T2 going to P by first turning RIGHT a to align its heading with the line and then going straight to P without any further turning. Up to now how do the two turnings compare?

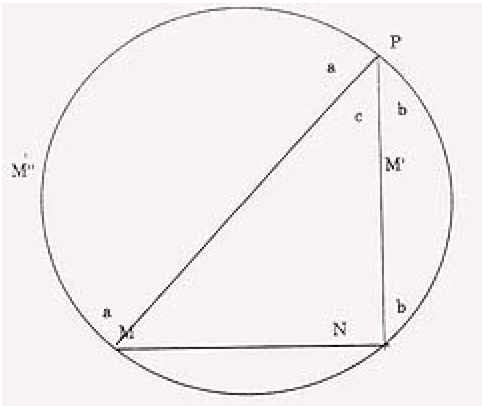


Figure 4

There are lots of ways to see that T1 turned exactly twice as much as T2. By symmetry you can see that the angle near P marked *a* is indeed the same as the angle near M. So you could say that T2 would have to do another RIGHT *a* to have the same heading as T1. So the turn in T1's trip was 2 x *a*. You can come to the same conclusion by noting that when Tl has traveled halfway to P it's heading is the same as T2's. In any case we conclude that *a* is half of the turn in the arc MM'P. Similarly, *b* must be half of the turn in the arc NN'P. So *a* + *b* is half of the turn in the whole arc MM'PN'N.

In other words *a* + *b* is the same for every position of P

Now we are getting there. We have an angle-related invariant! To pin down the angle *c*, look at the angles at P. Remembering that in TURTLE GEOMETRY we can talk about angles between lines and curves, we note that *a* + *b* + *c* is 180 degrees. So if *a* + *b* stays fixed, *c* must be as well.

6a. The Turtle as Connection Agent

I see this way of looking at the subtended angle theorem as making a closer connection between the entities involved -- a circle and the vertex of a triangle. There are many other connections that flow from it. As a pointer to start the interested reader's mind going, consider the following propositions:

Pl: The external angle of a triangle is equal to the sum of the opposite internal angles.

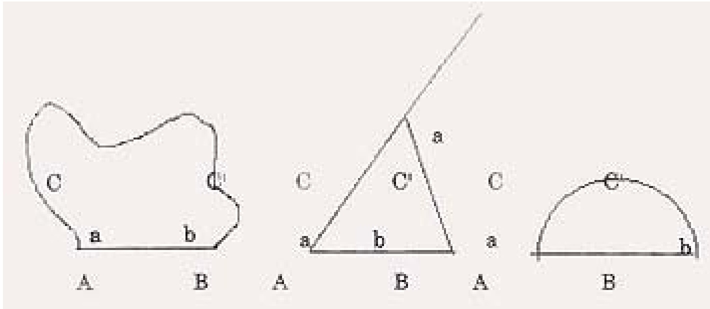


Figure 5

P2: The angle between a chord and a tangent is equal to the angle subtended in the opposite arc.

Are they related? Look at P3:

P3: Let AB be a straight line segment and ACC'B any curve that does not cross itself or AB. Then the turn in ACC'B is equal to the sum of the angles *a* and *b* between the curve and AB.

P1 is P3 with the triangle as C. P2 is P3 with the circular arc as C.

7. Conclusions: How Radical is *z*?

In the description of what I would put into *z* I have left to the end a discussion of which parts of SME I would leave out. In this conclusion I explore the question of whether something on the lines of *z* could satisfy the requirement mentioned in the opening section:

Could something be called "mathematics education" if it paid no attention to skill in numerical computation? Is it impossible to construct a mathematical education the set of topics of which is quite disjoint from school math and which would nonetheless serve all mathematically and socially defensible purposes claimed for learning mathematics? But how can we really know without investing time, effort and mathematical creativity into constructing counter-examples to the conservative (negative) answers?

Answering such questions might be far easier if somewhere there existed a consensus or even a prestigious body of discussion on the decision procedure for something to "serve all mathematically and socially defensible purposes claimed for learning mathematics." With the prospect of new opportunities for change it is becoming more important to find some way to deal with innovations that go beyond incremental changes that can be tested in the context of SME. However, not only is there no consensus on record; in fact I do not believe that such a definition of the goals of mathematics education is a meaningful concept. It is far too static in nature. As an alternative I propose a procedure in the form of a game called *Math Wars*.

To enter the game a player proposes an ame. The other players then raise questions about how this ame achieves some purpose; the original player answers or modifies the entry and so the dialogue proceeds in a Lakatos-like fashion. This is vague, but if the game were to be taken seriously by a segment of the mathematics education community, roles about the format of the entries and rules of engagement for the discussion would soon emerge in a process that would be as valuable as playing the game. More important: I conjecture that the process would converge both on a set of ames and on a more refined culture of discussion of the purposes of mathematics education.

Making a first entry I shall assume my own rules and simulate the beginnings of the kind of dialogue I think would be valuable.

My entry is defined as follows:

1. The strands of *z* I have mentioned together with the set of principles is an approximate definition of what would be included.
2. Certain parts of SME are definitely excluded: all the training directed at skills in manipulating multi-digit numbers, fractions and so on; most of school algebra.
3. Anticipation of criticisms, to which I'll now proceed.

7a. A Sample of Dialogue

Critic: This will leave citizens without skills that are essential to a complex society.

SP: There is no evidence that the skills learned with such effort are actually used and even if they are, mechanical devices can be substituted for them.

Critic: But you don't want to be dependent on your computer even if it is reduced to the size of a wristwatch.

SP: Inconsistency. You don't mind being dependent on your watch.

Critic: Anyway, the point of learning algebra and fractions is not to use it but to acquire what you would call powerful ideas.

SP: Like what?

Critic: Like variable, for example.

SP: In *z* children learn to use variables[11] as very empowering devices as early as the first two grades. In the context of writing programs the idea is concrete, powerful and assimilable.

Critic: Besides, the essential part of algebra is problem-solving and mathematizing.

SP: The fact that X produces the admittedly good effect Y is only an argument in favor of X if it can be shown that X is a particularly good way to produce Y. And this requires considering alternatives to X, which the algebra word problem researchers do at best very timidly. In my view experience has shown that the algebra word problems of the school curriculum are particularly bad entry routes to the skills you mention.

Critic: But some excellent research is beginning to show how to get better results in teaching word problems.

SP: I understand that this has become a challenge that the researchers in mathematics education do not want to abandon. But ego trips apart I believe that a better kind of research would be to find alternative mathematical activities rather than keep banging your head on these stone walls that are intrinsically poor anyway. Instead of making kids learn math let's make math kids will learn.

Critic: What's poor about what you call SME activities?

SP: They deal with boring problems rather than with personally engaging, long-term projects.

Critic: Word problems may not be serious projects but they teach the elements of making a mathematical model, using variables and even understanding language.

SP: Badly. And besides, Kafai's fourth graders did more and better mathematizing and much more problem-solving in making video games than all the word problems in all the algebra texts. And the extension of this activity in *z* based on ideas like object-based theories and greater access to better technology will make the disparity orders of magnitude greater.

Critic: But wait a moment. Research has shown that students have trouble with reification, so they will have even greater difficulty with *z* than with algebra.

SP: Your argument is an example of a typical vicious circle built into contemporary mathematics education research. If in fact students have trouble with reification this is because we have immersed them in an ame in which operations prime (and by a long shot) over objects. We mistake the results of our own training (successful for once in this case) for laws of psychology. The same is true of research that shows how bad people are at thinking probabilistically. School taught them to think in those ways (or at least reinforced any natural tendencies they had before going to school).

Critic: Anyway, if you think you have some good stuff why not propose to include it in school math? Accentuate the positive! Why do you need to be negative about what's already there?

SP: Two reasons. A superficial, but still very important, reason is time. Adding fragments students do not have the time to assimilate properly does more harm than good. The deeper reason is what Bateson calls deuterolearning: every time you learn something you learn two things, the other being the model of learning you just used. The stuff in SME carries with it a thoroughly bad model of mathematics and an even worse one of how to learn it. Admittedly the research community is doing a heroic job of jiggering SME here and there. But what is wrong is not a cold but a cancer. It has to be extirpated.

Critic (parting shot): Anyway your *z* is far less worked out than SME, which has been elaborated and polished by many generations of smart people. How can anything you make with the aid of a few friends be considered as a rival?

SP: You have a point.

ACKNOWLEDGMENTS

For advice and patience in preparing this paper I am deeply in debt to Uri Leron and Richard Noss. I had very helpful comments from Mitchel Resnick, Carol Sperry, Brian Silverman and an anonymous referee. Since the ideas accumulated over a long time too many people contributed to them for me to mention, but I want to single out Andrea diSessa, Alan Kay and Richard Noss and Brian Silverman. During the recent time when I thought most about probability Uri Wilensky was so close to me that I find it hard to be sure which ideas on this topic were worked out with him or even simply borrowed from him.

FOOTNOTES

1. And is one in the more liberal sense used in computer science, that is, any procedure that outputs a (not necessarily unique) computational object.

2. Logo specialists: Don't be put off by my modification to RANDOM to allow both lists and numbers as input. Traditionalists: Don't be put off by Logo's unorthodox functional notation. To connect with traditional function notation note that Logo will accept SETCOLOR (RANDOM ([blue green])) or even *g(f(x))* after definitions that make *f* equivalent to RANDOM and *g* to SETCOLOR. Both sides: these issues are relatively superficial anyway.

3. This is Logo's way of saying, "Take an integer less than 10,000 and append to it two zeros on the right."

4. The design of dynamic geometry programs is usually flawed in this respect, probably because of a deep-seated antagonism to anything that suggests programming as a mode of using the software. They consequently lost the chance to present the thing-like aspect of geometric operations performed by pull down, click and drag. Admittedly they enhance the thingness of the geometric objects, a good thing, even if this is probably the mathematical area where it is least needed.

5. Unpublished. I have a copy of the software, have worked with children with it and will describe it in my forthcoming book, *The Connected Family: Bridging the Digital Generation Gap.*

6. The multi-processing object-oriented nature of this manner of programming captures the idea of a superposition of two processes rather than of a single more contorted process. Thus it supports another powerful idea - superposition. It also supports a kind of reification to be discussed later.

7. Aaron Brandes, a doctoral candidate working under my supervision at MIT, has developed the idea that a group of children sometimes collectively generates intellectual steps characteristic of adult scientists that are not taken by individual children in the group. The group of children is more of a scientist than is the individual child.

8. The choice between ready-made tools and programmable systems is a classical example of a "problem" that needs to be dissolved rather than solved. But for costs, laziness and prejudices we would have systems with both features. However it is to be noted that using a graphing tool for the graphs in Figure 1 would be silly: the graphing side of the Logo programming consisted of three lines of code and ten minutes of debugging and made something flexible enough for an even smaller programming effort to produce the forced perturbations which probably would have needed more ingenuity to obtain, if it could be done at all, with a ready-made graphing tool.

9. Yasmin Kafai (1995) has extensively documented the feasibility of programming video-style games as an activity for 4^{th}- and 5^{th} -grade students under conditions far less favorable than those envisaged for *z*. In less formal studies I have worked with younger children with better software and a more supportive environment. Needless to say Kafai's findings are confirmed and exceeded.

10. During the refereeing process of this paper, I became aware of a tendency to misunderstand my intention. It was pointed out to me as problematic for the position I am arguing that there is a movement, perhaps most marked in the UK, to introduce experiences of motion in elementary school. But my point is not that school does not try to talk about motion but that there is no conceptual framework for dealing with it in a non-superficial way. Consider, for example, the parabolic path of a thrown stone. Checking through

science texts I find that most make an authoritarian declaration that it is so. Some describe an experiment to find out. None give any insight into why it should be so. I don't blame them. The only way I know to do it requires a shift in the material conditions of school from primarily paper to primarily dynamic media and no less in the conceptual dimensions.

11. See Noss (in press) for a discussion of acquiring the concept of variable in the context of Logo programming.

REFERENCES

Abelson, H. and diSessa, A. (1980). *Turtle Geometry: The Computer as a Medium for Exploring Mathematics.* Cambridge, MA: MIT Press.

Clayson, J. (1990). *Visual Modeling with Logo: A Structural Approach to Seeing.* Cambridge, MA: MIT Press.

Confrey, J. and Costa, S. (In press). A critique of the selection of "Mathematical Objects" is central metaphor for advanced mathematical thinking. *International Journal of Computers for Mathematical Learning.*

diSessa, A., Holyes, C., Noss, R. and Edwards, L. (Eds.) (1996). *Computers and Exploratory Learning.* Heidelberg: Springer Verlag

Dorfler, W. (1993). Computer use and views of the mind. In C. Keitel and K. Ruthven (Eds.), *Learning from Computers: Mathematics Education and Technology* (pp. 159-186). New York: Springer Verlag.

Dubinsky, E. and Harel, G. (1992). The nature of the process conception of function. In G. Harel and E. Dubinsky (Eds.), *The Concept of Function.* Washington, DC: Mathematical Association of America.

Greene, Andree. Logo Art. Shown in art exhibitions in Montreal and elsewhere.

Harel, Idit (1991). *Children Designers.* NJ: Ablex.

Hoyles, C. and Noss, R. (Eds.) (1992). *Learning Mathematics and Logo.* Cambridge, MA: MIT Press.

Kafai, Y. (1995). *Minds In Play: Computer Game Design as a Context for Children's Learning.* NJ: Erlbaum.

Kahneman, D. and Tversky, A. (1982). *Judgment Under Uncertainty.* Cambridge: Cambridge University Press.

Kaput, J. J. (1992). Technology and Mathematics Education. In A. C. Grouws (Ed.), *Handbook of Research and Teaching in Mathematics.* New York: Macmillan.

Lawler, R. (1981). The progressive construction of mind. *Cognitive Science*, 5: 1-30. < p>Lawler, R. (1985). *Computer Experience and Cognitive Development.* NY: Wiley.

Noss, R. and Hoyles, C. (1996). *Windows on Mathematical Meaning: Learning Cultures and Computers.* Dordrecht: Kluwer.

Noss, R. (In press). On static and dynamic algebra systems. In J. Kaput (Ed.), *Early Algebra.*

Papert, S. (1980). *Mindstorms: Children, Computers and Powerful Ideas.* New York: Basic Books, Inc.

Papert, S. (1986). Beyond the cognitive: The other face of mathematics. Proceedings of the Nineteenth International Conference for the Psychology of Mathematics Education, Plenary Lectures, 51-58.

Papert, S. (1992).*The Children's Machine: Rethinking School in the Age of the Computer.* New York: Basic Books, Inc.

Papert, S. (In press) *The Connected Family: Bridging the Digital Generation Gap.* Atlanta: Longstreet Press, Inc.

Resnick, M. (1996). New paradigms for computing, new paradigms for thinking. In A. diSessa, C. Hoyles, R. Noss, and L. Edwards (Eds.), *The Design of Computation Media to Support Exploratory Learning.* (pp. 31-44).

Resnick, M. (1994). *Turtles, Termites, and Traffic Jams: Explorations in Massively Parallel Microworlds.* Cambridge, MA: MIT Press.

Resnick, M., Ocko, S. and Papert, S. (1988). LEDO, logo, and design. *Children's Environments Quarterly,* 5(4): 14-18.

Sfard, A. and Leron, U. (1996). Just give me a computer and I will move the earth: Programming as a catalyst of a cultural revolution in the mathematics classroom. *International Journal of Computers for Mathematical Learning.*

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In G. Harel and E. Dubinsky (Eds.), *The Concept of Function* (pp. 59-84). Washington, DC: Mathematical Association of America.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics,*14(1): 44-55.

Shaughnessy, J. M. (1992). Research in probability and statistics. In A. G. Grouws (Ed.), *Handbook of Research and Teaching in Mathematics.* New York: Macmillan.

Solomon, C. *Computer Environments for Children: A Reflection on Theories of Learning.* Cambridge MA: MIT Press.

Weir,S. (1987). *Cultivating Minds: A Logo Casebook.* New York: Harper & Row.

Wilensky, U. (1996). Modeling Rugby: Kick First, Generalize Later? *International Journal of Computers for Mathematical Learning,* 1(1).

Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematics education. In I. Harel and S. Papert (Eds.), *Constructionism*. Norwood, NJ: Ablex Publishing Corp.

Wilensky, U. (1993). *Connected Mathematics: Building Concrete Relationships with Mathematical Knowledge.* Doctoral dissertation. Cambridge, MA: MIT Media Laboratory.

Wilensky, U. (1995). Paradox, programming and learning probability. *Journal of Mathematical Behavior.* 14(2).